Comparing the Core and Shapley Value in an Example

Game Theory Course:
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Compare Core and Shapley Value in an Example

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  - 10 temporary members
  - 5 permanent members can veto resolutions.

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- UN security council: represent it as a cooperative game.

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- UN security council: represent it as a cooperative game.
  - China, France, Russia, UK, US are labeled $\{1, 2, 3, 4, 5\}$
  - $v(S) = 1$ if $\{1, 2, 3, 4, 5\} \subset S$ and $\#S \geq 8$,
  - $v(S) = 0$ otherwise.

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  - Core: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.

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    - 1's value: $v(\{1,2,3\}) - v(\{2,3\}) = 1$ weighted by 2/6, $v(\{1,2\}) - v(\{2\}) = 1$ weighted by 1/6, $v(\{1,3\}) - v(\{3\}) = 1$ weighted by 1/6

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    - 2's value: $v(\{1,2\}) - v(\{1\}) = 1$, weighted by 1/6

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    - 3's value: $v(\{1,3\}) - v(\{1\}) = 1$, weighted by 1/6

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- $v(S) = 1$ if $1 \in S$ and $\#S \geq 2$, $v(S) = 0$ otherwise.
  - Shapley Value$_i = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[v(S \cup \{i\}) - v(S)\Big]$.
    - 1's value: $v(\{1,2,3\}) - v(\{2,3\}) = 1$ weighted by 2/6, $v(\{1,2\}) - v(\{2\}) = 1$ weighted by 1/6, $v(\{1,3\}) - v(\{3\}) = 1$ weighted by 1/6
    - 2's value: $v(\{1,2\}) - v(\{1\}) = 1$, weighted by 1/6
    - 3's value: $v(\{1,3\}) - v(\{1\}) = 1$, weighted by 1/6
  - Shapley Value: $x_1 = 2/3$, $x_2 = 1/6$, $x_3 = 1/6$.

A way to the Shapley Value:

Cooperative Games

- Model complex multilateral bargaining and coalition formation, without specifying the particulars of a normal or extensive form

  - Core: Based on coalitional threats - each coalition must get at least what it can generate alone

  - Shapley Value: based on marginal contributions: what does each player contribute to each possible coalition.

  - Other solutions…